Problem Set #9

1. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-y}}{y} \qquad \text{for } 0 < x < y,\ 0 < y < \infty$$

Compute $E(X^3|Y = y)$.

$$g(X|Y) = \frac{f(x,y)}{f_Y(y)}$$

$$\begin{aligned} f_Y(y) &= \int_0^y \frac{e^{-y}}{y} dx \\ &= \frac{e^{-y}}{y} \int_0^y dx \\ &= \frac{e^{-y}}{y} (x|_0^y) \\ &= \frac{e^{-y}}{y} (y - 0) \\ &= e^{-y} \end{aligned}$$

$$\begin{aligned} g(X|Y) &= \frac{\frac{e^{-y}}{y}}{e^{-y}} \\ &= \frac{1}{y} \end{aligned}$$

$$\begin{aligned} E(X^3|Y &= y) = \int_0^y x^3 \frac{1}{y} dx \\ &= \frac{1}{y} \int_0^y x^3 dx \\ &= \frac{1}{y} \left(\frac{1}{4} x^4 |_0^y \right) \\ &= \frac{1}{4} y^3 \end{aligned}$$

2. A prisoner is trapped in a cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 2 days' travel. The second leads to a tunnel that returns him to his cell after 4 days' travel. The third door leads to freedom after 1 day of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3, with the respective probabilities 0.5, 0.3, and 0.2, what is the expected number of days until the prisoner reaches freedom?

Let X denote the number of days until the prisoner is free, and let M denote the initial door chosen. Then

$$\begin{aligned} E[X] &= E[X|I=1](.5) + E[X|I=2](.3) + E[X|I=3](.2) \\ &= (2+E[X])(.5) + (4+E[X])(.3) + .2 \\ &= 12 \end{aligned}$$

3. Compute the variance in the length of time until the prisoner reaches freedom.

$$\begin{aligned} V(X) &= E\left[V(X|I)\right] + V\left[E(X|I)\right] \\ &= 0.5\sigma_1^2 + 0.3\sigma_2^2 + 0.2\sigma_3^2 + 0.5(2+E[X]-12)^2 + 0.3(4+E[X]-12)^2 + 0.2(1-12)^2 \\ &= 0.5\sigma_1^2 + 0.3\sigma_2^2 + 0.2\sigma_3^2 + 0.5(4) + 0.3(16) + 0.2(-11)^2 \end{aligned}$$

4. Type i light bulbs function for a random amount of time having mean μ_i and standard deviation of σ_i, for $i = 1, 2$. A light bulb randomly chosen from a bin of bulbs is a type 1 bulb with probability p and a type 2 bulb with probability $1-p$. Find the expected value and variance of the lifetime of this randomly chosen bulb.

$$\begin{aligned} E[X] &= E[X|\text{type } 1]p + E[X|\text{type } 2](1-p) \\ &= p\mu_1 + (1-p)\mu_2 \end{aligned}$$

Let I be the type.

$$\begin{aligned} E[X|I] &= \mu_I \\ V(X|I) &= \sigma_I^2 \end{aligned}$$

$$\begin{aligned} V(X) &= E[\sigma_I^2] + V(\mu_I) \\ &= p\sigma_1^2 + (1-p)\sigma_2^2 + p\mu_1^2 + (1-p)\mu_2^2 - [p\mu_1 + (1-p)\mu_2]^2 \end{aligned}$$