# Answers to Problem Set

1) Denote the bivariate probability density function (*population*) for X and Y by f(x,y). Let f(x,y) equal x + y for $\{0 \le X \le 1\}$ and $\{0 \le Y \le 1\}$, zero otherwise. Find the marginal distributions for X and Y (*population*), respectively. Please denote them by $f_x(x)$ and $f_Y(y)$.

The marginal distribution for X equals $\int_0^1 f(x,y)\,dy$. Therefore, $f_x(x) = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)\Big|_0^1 = \left(x + \frac{1}{2}\right)$ for $\{0 \le X \le 1\}$, zero otherwise. By symmetry, $f_y(y) = \left(y + \frac{1}{2}\right)$ for $\{0 \le Y \le 1\}$, zero otherwise.

2) Using the same bivariate probability density function (*population*), find the covariance (*parameter*) between X and Y.

For covariance requires the following moments: E(XY), E(X), and E(Y).

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dy\,dx = \int_0^1\int_0^1 \left(x^2y + y^2x\right)dy\,dx = \int_0^1 \left[\left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)\Big|_0^1\right]dx =$$

$$\int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right)dx = \left(\frac{x^3}{6} + \frac{x^2}{6}\right)\Big|_0^1 = \frac{1}{3}.$$

$$E(X) = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)\Big|_0^1 = \frac{7}{12}.$$

E(Y) also equals $\frac{7}{12}$ by symmetry.

Therefore, $C(X,Y) = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$, by the covariance and variance theorem.

3) Continuing with the same definitions of X and Y, define $W_1 = 6 + 12X + 24Y$. Find the variance (*parameter*) of $W_1$.

If W = a + bX + cY, $V(W) = b^2V(X) + c^2V(Y) + 2bcC(X,Y)$ by the theorem for linear functions. From the previous problem, $C(X,Y) = -\frac{1}{144}$ and $E(X) = E(Y) = \frac{7}{12}$. To find the variance of X and the variance of Y, you additionally need $E(X^2)$ and $E(Y^2)$.

$$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \int_0^1\left(x^3+\frac{x^2}{2}\right)dx = \left.\left(\frac{x^4}{4}+\frac{x^3}{6}\right)\right|_0^1 = \frac{5}{12}.$$

$E(Y^2) = \frac{5}{12}$ by symmetry. Therefore, $V(Y) = V(X) = \frac{5}{12}-\left(\frac{7}{12}\right)^2 = \frac{11}{144}$.

If $W_1 = 6 + 12X + 24Y$, then $V(W_1) = 12^2\frac{11}{144}+24^2\frac{11}{144}+2\cdot 12\cdot 24\left(-\frac{1}{144}\right) = 51$.

4) Define $W_2 = 12 + 12X - 24Y$. Find the covariance (*parameter*) of $W_2$.

If $W_2 = 12 + 12X - 24Y$, then $V(W_2) = 12^2\frac{11}{144}+24^2\frac{11}{144}+2\cdot 12\cdot(-24)\left(-\frac{1}{144}\right) = 59$.

5) Find the covariance (*parameter*) between $W_1$ and $W_2$.

If $W_1 = a_1 + b_1X + c_1Y$ and $W_2 = a_2 + b_2X + c_2Y$, then $C(W_1,W_2) = b_1b_2V(X) + c_1c_2V(Y) + (b_1c_2 + b_2c_1)C(X,Y)$ by the theorem for a pair of linear functions.

$$C(W_1,W_2) = 12^2\frac{11}{144}+24\cdot(-24)\frac{11}{144}+\left[12\cdot(-24)+12\cdot 24\right]\left(-\frac{1}{144}\right) = -33.$$

6) Find the correlation coefficient (*parameter*) between $W_1$ and $W_2$.

$$\rho_{w_1w_2} = \frac{C(W_1,W_2)}{\sqrt{V(W_1)V(W_2)}} = \frac{-33}{\sqrt{51\times 59}} = -0.6016.$$